

Exercises for 'Topics in complex analysis'

(10/09/2025)

H 1.1 (Does it exist or not?)

Give an example of the following objects or prove that they cannot exist.

- (i) A holomorphic function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ such that $f'(z) = \frac{1}{z}$.
- (ii) A holomorphic function $f : \mathbb{C} \setminus \{\operatorname{Re}(z) \leq 0\} \rightarrow \mathbb{C}$ such that $f'(z) = \frac{1}{z}$.
- (iii) A domain $D \subset \mathbb{C}$ and a non-constant holomorphic function $f : D \rightarrow \mathbb{C}$ such that $|f|$ is constant.
- (iv) A non-constant holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f(n) = 0$ for all $n \in \mathbb{N}$.
- (v) A non-constant holomorphic function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $f(\frac{1}{n} + i) = 0$ for all $n \in \mathbb{N}$.
- (vi) A non-constant holomorphic function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ that is bounded.

H 1.2 (On the image of entire functions)

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant entire function (i.e. f is holomorphic on the whole complex plane). Show that its image $f(\mathbb{C})$ is dense in \mathbb{C} .

Hint: Argue by contradiction and use Liouville's theorem for a suitable function.

H 1.3 (The Arzelà-Ascoli theorem on \mathbb{C})

Let $K \subset \mathbb{C}$ be a compact set and let $f_n : K \rightarrow \mathbb{C}$ be an equibounded sequence of functions, i.e., there exists a constant $M > 0$ such that

$$\sup_{z \in K} |f_n(z)| \leq M \quad \forall n \in \mathbb{N}.$$

a) Let $S \subset K$ be a countable set. Show that there exists a subsequence f_{n_k} such that $f_{n_k}(z)$ converges to some value $f_z \in \mathbb{C}$ for all $z \in S$.

b) Now assume in addition that the sequence $f_n : K \rightarrow \mathbb{C}$ is equicontinuous, i.e., for all $\varepsilon > 0$ and $z \in K$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ we have the implication

$$|y - x| < \delta \quad \Rightarrow \quad |f_n(y) - f_n(x)| < \varepsilon.$$

Show that if S is dense in K , then $f_{n_k}(z)$ converges to some $f_z \in \mathbb{C}$ for all $z \in K$, where f_{n_k} denotes the subsequence from item a).

c) Under the assumptions of b), define the function $f : K \rightarrow \mathbb{C}$ by $f(z) := f_z$. Show that $f_{n_k} \rightarrow f$ uniformly on K and conclude that f is continuous.